

THE ACTION OF THE CREMONA GROUP ON THE NON-COMMUTATIVE RING.

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ABSTRACT. The Cremona group acts on the field of two independent commutative variables over complex numbers. We provide a non-commutative ring that is an analog of non-commutative field of two independent variables and prove that the Cremona group embeds in the group of outer automorphisms of this ring. First proof of this result is technical, the second one is conceptual and proceeds through a construction of a birational invariant of an algebraic variety from its bounded derived categories of coherent sheaves.

1. INTRODUCTION

The Cremona group is the group of automorphisms of the field $K = \mathbf{C}(x, y)$ of two independent commuting variables. Alternatively it is the group of birational automorphisms of \mathbf{CP}^2 or any other rational surface, because K is its field of rational functions.

We construct a non-commutative associative algebra A , which is a consecutive localization of a free non-commutative algebra of two generators at the set of elements outside a commutator ideal. The main result is that the Cremona group acts on this non-commutative algebra by outer automorphisms. First proof of this statement is a direct computation: we provide non-commutative analogs of generators of Cremona group and verify that the relations known in commutative setting also hold in a non-commutative up to inner automorphisms.

Then we prove the same fact more abstractly, namely for a smooth proper algebraic variety X we construct a triangulated category $\tilde{C}(X)$, which is a birational invariant of X . By this we mean that if X and Y are birationally equivalent, then $\tilde{C}(X)$ and $\tilde{C}(Y)$ are equivalent as triangulated categories. The category $\tilde{C}(X)$ is a quotient of derived category of coherent sheaves that are left orthogonal to the structure sheaf of O_X by the full subcategory of complexes of sheaves with the support of codimension at least 1. Then we study the structure of $\tilde{C}(\mathbf{CP}^2)$ and prove that it is generated by one object P which is the image of $O(1)$. We also check that this object is preserved under equivalences, so the Cremona group will act by outer automorphisms on the ring of endomorphisms of this object. Then we compute that $RHom_{\tilde{C}(\mathbf{CP}^2)}^0(P, P) = A$, so we get an action of the Cremona group on the same non-commutative algebra as previous. Then we verify that two

actions coincide.

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2. CONSTRUCTION OF THE NONCOMMUTATIVE ALGEBRA A

In what follows we will construct a non-commutative algebra A together with algebra homomorphism $\phi : A \rightarrow K$ such that:

1. $R = \mathbf{C} \langle x, y \rangle \subset A$, R is an algebra of polynomials in two non-commuting variables.
2. $I = A[A, A]A$ is the kernel of ϕ
3. All the elements in $A \setminus I$ are invertible

Let us construct such an algebra by induction. Put $A_0 := R = \mathbf{C} \langle x, y \rangle$, $I_0 := R[R, R]R = R(xy - yx)R$ (just notice that for associative noncommutative algebras B the following holds $B[B, B]B = [B, B]B = B[B, B]$ and we call this submodule a commutator of B). We successively construct A_{i+1} as a localization of A_i at $A_i \setminus I_i$. I_i is just a commutator of A_i , denote $S_i = A_i \setminus I_i$. First let us explain what do we mean by localization: consider $B_i := A_i \langle u_s | s \in S_i \rangle$ and let P_i be a bi-ideal in B_i generated by $u_s s - 1$ and $s u_s - 1$. We put $A_{i+1} := B_i / P_i$. We also have induced maps $A_i \rightarrow A_{i+1}$, so we may take an inductive limit $A := \varinjlim A_i$.

We verify the following:

Lemma 2.1. (1). $\phi_i : A_i \rightarrow K$ is well defined and consistent with the maps $A_i \rightarrow A_{i+1}$.
 (2). $\ker(\phi_i) = I_i$.

Corollary. $\phi : A \rightarrow K$ is well defined and $\ker(\phi) = I$.

We can define ϕ on the inductive limit of A_k 's which is A and the statement $\ker(\phi) = I$ follows from the corresponding statement about ϕ_k .

Proof. For $i = 0$ we define an obvious map $\phi_0 : R \rightarrow \mathbf{C}[x, y] \subset K$. It has the commutator I_0 as the kernel.

By induction we suppose that $\ker(\phi_i) = I_i$ for $i = k$. Then ϕ_k maps S_k to non-zero thus invertible elements of K , so we may extend the map $\phi_k(u_s) := \phi_k(s)^{-1}$ to B_k and it is easily seen that P_k maps to 0, so we may define ϕ_{k+1} in a consistent way on $A_{k+1} = B_k / P_k$.

So S_k may be characterized as the set of elements that maps by ϕ_k not to zero, so it is a multiplicative system. We may verify the following equality in A_{k+1} :

$u_s u_t = u_{ts}$, because $ts(u_{ts} - u_s u_t) = (tsu_{ts} - 1) + t(su_s - 1)u_t + (tu_t - 1) \in P_k$. From the other side $(u_{ts}ts - 1)(u_{ts} - u_s u_t) \in P_k$. It follows that $(u_{ts} - u_s u_t) \in$

P_k .

Let us prove now that $\ker(\phi_{k+1}) = I_{k+1}$ by induction on k . Commutator of A_{k+1} is contained in the kernel of ϕ_{k+1} , because K is commutative, so we have to prove another inclusion. For any element $b \in B_k$ there exist $a, s \in A_k$ such that $b - au_s \in B_k[B_k, B_k] + P_k$, because

$$a_1 u_{s_1} a_2 u_{s_2} \dots a_n - a_1 a_2 \dots a_n u_{s_1} u_{s_2} \dots u_{s_{n-1}} \in B_k[B_k, B_k]$$

$$au_{s_1} u_{s_2} \dots u_{s_k} - au_{s_k} \dots u_{s_2} s_1 \in P_k$$

$$au_s + bu_p - (ap + bs)u_{ps} \in B_k[B_k, B_k] + P_k$$

In particular for any $b' \in A_{k+1} = B_k/P_k$ there exist $a', s' \in A_k$, such that $b' - a'u'_s \in A_{k+1}[A_{k+1}, A_{k+1}] = I_{k+1}$. So if $\phi_{k+1}(b') = 0$ then $\phi_{k+1}(a'u'_s) = \phi_k(a')\phi_k(s')^{-1} = 0$, so $\phi_k(a') = 0$ and by induction hypothesis $a' \in A_k[A_k, A_k] = I_k$. This proves that $b' \in I_{k+1}$. \square

To define a morphism from an algebra A to any algebra B we should prescribe images of x, y . Then a map from A_0 to B will be defined. If the image of elements not in commutator of A_k is invertible in B , then we can extend it to the map from A_{k+1} to B .

Lemma 2.2. *Suppose $a, b \in A$ such that the field morphism $i : K \rightarrow K$ given by $i : x \mapsto \phi(a)$, $i : y \mapsto \phi(b)$ is injective. Then there is a well defined algebra homomorphism $f : A \rightarrow A$ with $f(x) = a$, $f(y) = b$.*

Proof. Given $a, b \in A$ we uniquely define $f_0 : A_0 \rightarrow A$ by prescribing $f_0(x) = a$, $f_0(y) = b$. Now by induction suppose that the map $f_k : A_k \rightarrow A$ is well defined and $f_k(x) = a$, $f_k(y) = b$. Now we use $\phi \circ f_k = i \circ \phi_k$. It follows that $\phi(f(S_k))$ is non-zero, because $\phi_k(S_k)$ is non-zero and i is injective. So $f_k(S_k)$ is invertible and we can uniquely extend f_k to $f_{k+1} : A_{k+1} \rightarrow A$. \square

To see that A is indeed non-commutative and has non-trivial elements (it's not obvious from the construction that A doesn't coincide with K) let us construct algebra morphisms from A to the following algebra $M = M_{k \times k}(K[[\epsilon]])$ - algebra of matrices over $K[[\epsilon]]$ series with coefficients in K . Let us choose two matrices of the form $X = xId + \epsilon X_1$, $Y = yId + \epsilon Y_1$, where X_1, Y_1 are some matrices in M . We can invert element of M if and only if we can invert it's reduction modulo ϵ . This justifies that the map $R \rightarrow M$ given by $x \mapsto X$, $y \mapsto Y$ may be extended to the morphism $A \rightarrow M$.

For example we see that $XY - YX = \epsilon^2(X_1 Y_1 - Y_1 X_1)$ may be non-zero, implying that $xy - yx$ is nonzero in A , so A is indeed non-commutative.

Lemma 2.3. *The natural map $i : A_0 = \mathbf{C} \langle x, y \rangle \rightarrow A$ is an injection.*

Proof. Let us consider an algebra $M = M_{k \times k}(K[[\epsilon]])$ of matrices of big enough size k with entries in the ring of formal power series $K[[\epsilon]]$. For any pair of complex matrices $S, T \in M_{k \times k}(\mathbf{C})$ it exists a ring homomorphism $f : A \rightarrow M$ given by $x \mapsto xId + \epsilon S$, $y \mapsto yId + \epsilon T$. So if some non-commutative polynomial $a \in A_0$ was in the kernel of the map $i : A_0 \rightarrow A$,

then $f \circ i(a) = 0$ and it's a polynomial in ϵ with coefficients in matrices, in particular for the highest power of ϵ we would have $a(S, T) = 0$. So we would have that some non-commutative polynomial is identically 0 if we put as arguments any complex matrices of any size. This is false, so we proved the lemma. \square

3. DIRECT EXTENSION OF THE CREMONA GROUP TO NON-COMMUTATIVE VARIABLES

Suppose $a = \begin{pmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{pmatrix} \in PGL(2, k(x))$. It gives an automorphism of K over \mathbf{C} :

$$a : (x, y) \mapsto (x, \frac{yP(x) + Q(x)}{yR(x) + S(x)})$$

Define also a map $\tau : (x, y) \mapsto (y, x)$.

Let us summarize the results of [3]:

The Cremona group Cr is generated by τ and $PGL(2, k(x))$ with the described action.

Consider an exact sequence of groups $1 \rightarrow PGL(2, k(x)) \rightarrow B \rightarrow PGL(2, k) \rightarrow 1$, where B preserves the subfield $k(x) \subset k(x, y)$. $D = PGL(2, k) \times PGL(2, k) = D_1 \times D_2 \subset B$ preserves both $k(x)$ and $k(y)$, where the action of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_1$ is given by $(x, y) \mapsto (\frac{ax+b}{cx+d}, y)$. And the action of $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_2$ is given by $(x, y) \mapsto (x, \frac{ay+b}{cy+d})$.

The Cremona group is generated by τ, B and by [3] all the relations are the consequences of the following:

1. $\tau^2 = 1$
2. $\tau D \tau = D$
3. $(\tau e)^3 = a$

in the last relation $e : (x, y) \mapsto (x, \frac{x}{y}) \in B$ and $a : (x, y) \mapsto (\frac{1}{x}, \frac{1}{y}) \in D$

Now let us reinterpret these relations: we denote $D_1 = D_2 = PGL(2, k)$ and $G = PGL(2, k(x))$. Field inclusion $k \subset k(x)$ induces group inclusion $D_2 = G(k) \subset G = G(k(x))$. D_1 acts on $k(x)$, so we have also the action of D_1 on coefficients of $PGL(2, k(x)) = G$ which leaves D_2 stable.

Consider the group generated by G and $\mathbf{Z}/2 = \langle 1, \tau \rangle$, on which we put the following relations:

1. $D_1 = \tau D_2 \tau$ normalizes G and the induced action of D_1 on G is induced from the action of $PGL(2, k) = D_1$ on $k(x)$.
2. $(\tau e)^3 = a$, where $e = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \in G$ and $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in D_1 \times D_2$.

The group B from previous notations is included in the exact sequence: $1 \rightarrow G \rightarrow B \rightarrow D_1 \rightarrow 1$. So the the group that we described is the Cremona group, as this set of relations is equivalent to the previous one.

Let us now introduce an action on the previously constructed ring A . $a \in GL(2, k(x))$ is a matrix $\begin{pmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{pmatrix}$. It gives automorphisms of A :

$$t_a : (x, y) \mapsto (x, (yR(x) + S(x))^{-1}(yP(x) + Q(x)))$$

$$p_a : (x, y) \mapsto (x, (P(x)y + Q(x))(R(x)y + S(x))^{-1})$$

Before proceeding further, let us show how those automorphisms are related: the dual algebra for $R = \mathbf{C} \langle x, y \rangle$ would be $R^{op} = \mathbf{C} \langle x, y \rangle$. They are isomorphic as algebras:

$$\begin{array}{ccc} \rho : & R & \longrightarrow & R^{op} \\ & (x, y) & \longmapsto & (x, y) \end{array}$$

In particular $\rho(xy) = yx$ and $\rho(x^2y^3x^7) = x^7y^3x^2$.

We can extend ρ as an isomorphism between A and A^{op} , so we would have $\rho(P^{-1}Q) = \rho(Q)\rho(P)^{-1}$. Now we observe that formulas for t_a and p_a are dual:

$$\rho \circ t_a = p_a \circ \rho$$

So the result we would prove for t_a would also be true for p_a .

Define a map $\tau : (x, y) \mapsto (y, x)$, and consider a group Cr^{nc} generated by τ and t_a for all $a \in GL(2, k(x))$. Commutator $A[A, A]A$ is preserved by any automorphism, so Cr^{nc} also acts on $A/A[A, A]A = \mathbf{C}(x, y)$ i.e. maps to commutative Cremona group Cr . Let us denote Cr^{in} the subgroup of Cr^{nc} generated by inner automorphisms, i.e. of the type $(x, y) \mapsto (rxr^{-1}, yry^{-1})$ for some invertible $r \in A^*$. We want to prove that Cr^{in} is the kernel of the map $Cr^{nc} \rightarrow Cr$.

Theorem 3.1. *The kernel of the map $Cr^{nc} \rightarrow Cr$ consists of inner automorphisms of A .*

Proof. For this we need to check that when we lift a relation for generators of Cr to Cr^{nc} it belongs to Cr^{in} .

Lemma 3.1. $t_a t_b = t_{ab}$ for any $a, b \in GL(2, k(x))$

Proof. Let $b = \begin{pmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{pmatrix}$, $a = \begin{pmatrix} P_1(x) & Q_1(x) \\ R_1(x) & S_1(x) \end{pmatrix}$. The action of a on A is given by

$$\begin{aligned} t_b : (x, y) &\mapsto (x, (yR(x) + S(x))^{-1}(yP(x) + Q(x))) =: (x, y_1) \\ t_a : (x, y_1) &\mapsto (x, (y_1R_1(x) + S_1(x))^{-1}(y_1P_1(x) + Q_1(x))) =: (x, y_2) \\ y_1R_1(x) + S_1(x) &= (yR + S)^{-1}(y(PR_1 + RS_1) + (QR_1 + SS_1)) \end{aligned}$$

$$y_1 P_1(x) + Q_1(x) = (yR + S)^{-1}(y(PP_1 + RQ_1) + (QP_1 + SQ_1))$$

$$t_a t_b : (x, y) \mapsto (x, (y(PR_1 + RS_1) + (QR_1 + SS_1))^{-1}(y(PP_1 + RQ_1) + (QP_1 + SQ_1)))$$

So $t_a t_b = t_{ab}$, because

$$ab = \begin{pmatrix} P_1(x)P(x) + Q_1(x)R(x) & P_1(x)Q(x) + Q_1(x)S(x) \\ R_1(x)P(x) + S_1(x)R(x) & R_1(x)Q(x) + S_1(x)S(x) \end{pmatrix}$$

□

When $GL(2, k(x)) \subset Cr^{nc}$ maps to Cr it has a center in the kernel, so suppose $d = \begin{pmatrix} d(x) & 0 \\ 0 & d(x) \end{pmatrix}$ is a diagonal matrix, then $t_d : (x, y) \mapsto (x, d(x)^{-1}yd(x))$ is a conjugation by $d(x)$, so $t_d \in Cr^{in}$.

Let us verify a first relation: suppose $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_2 = GL(2, k)$. Then $m_1 = \tau t_m \tau : (x, y) \mapsto ((cx + d)^{-1}(ax + b), y)$. From this formula we easily see that m_1 normalizes a subgroup $GL(2, k(x)) \subset Cr^{nc}$ and it's action is induced from it's action on $k(x)$, so the first relation is verified.

Let us now verify the second relation: $(\tau e)^3 = a$. Remind that

$$\tau : (x, y) \mapsto (y, x)$$

$$e : (x, y) \mapsto (x, y^{-1}x)$$

$$a : (x, y) \mapsto (x^{-1}, y^{-1})$$

Then we calculate

$$\tau e : (x, y) \mapsto (y^{-1}x, x)$$

$$(\tau e)^2 : (x, y) \mapsto (x^{-1}y^{-1}x, y^{-1}x)$$

$$a^{-1}(\tau e)^3 : (x, y) \mapsto (x^{-1}yxy^{-1}x, x^{-1}yx)$$

The last automorphism is a conjugation by $x^{-1}y$, so $a^{-1}(\tau e^3) \in Cr^{in}$. □

4. THE PROOF USING DERIVED CATEGORIES

In this section we would construct a triangulated category \tilde{C} with a specific isomorphism class of objects P , and it would be a birational invariant of rational surfaces. As a consequence, the Cremona group will act on this category by equivalences and will preserve P , so it will act by outer automorphisms on the graded ring $H^*(End_C(P))$, in particular it's degree 0 part coincides with the non-commutative algebra A considered previously. We would then verify that the action of Cr on A coincides with the one given by explicit formulas.

4.1. Notations. First let us make some conventional notations:

D - bounded derived category of coherent sheaves on \mathbf{CP}^2

$\pi : X \rightarrow \mathbf{CP}^2$ - sequence of blow-ups of \mathbf{CP}^2 or equivalently a regular map invertible outside a finite set of points.

$D(X)$ - bounded derived category of coherent sheaves on X . By omitting X we would mean the corresponding category for \mathbf{CP}^2 .

$O, O(1), O(2)$ - line bundles $O_{\mathbf{CP}^2}, O_{\mathbf{CP}^2}(1), O_{\mathbf{CP}^2}(2)$, we would also see them as objects of D placed at degree 0. Depending on the context we would denote by the same symbols the objects $L\pi^*O, L\pi^*O(1), L\pi^*O(2) \in D(X)$ for the functor $L\pi^* : D \rightarrow D(X)$.

E - exceptional curve of a blow-up.

$\tilde{D}(X)$ - full subcategory of $D(X)$ consisting of complexes of sheaves left orthogonal to O_X .

$D^1(X)$ - full subcategory of $D(X)$ of complexes of sheaves with the dimension of support ≤ 1 .

$\tilde{D}^1(X)$ - full subcategory of $D(X)$ of objects which are both in $\tilde{D}(X)$ and in $D^1(X)$.

$RHom(F, G)$ would usually be a derived functor in category $D(X)$. If both F, G belong to D , we consider $RHom$ in D , because $L\pi^*$ is a fully faithful functor.

$\tilde{C}(X) = \tilde{D}(X)/\tilde{D}^1(X)$ - quotient triangulated category.

$P \in \tilde{C}(X)$ - isomorphism class of object $O(1)$.

$a, b, e \in Hom(O, O(1))$ - sections corresponding to X, Y, Z .

$R = \mathbf{C} \langle x, y \rangle$ - ring of polynomials in non-commutative variables.

A - non-commutative algebra, constructed in previous sections.

$K = \mathbf{C}(x, y)$

$A^\bullet[1]$ a shift of a complex A^\bullet to the left

4.2. Strategy of the proof. First we prove that the quotient category $\tilde{C} = \tilde{D}/\tilde{D}^1$ is split generated by one object P and $RHom_{\tilde{C}}(P, P) \cong A^\bullet$ is some dg-algebra. We prove that this dg-algebra is concentrated in non-positive degrees and $H^0(A^\bullet) = A$ is a non-commutative algebra that we constructed earlier.

Next we prove that the functor $L\pi^* : D \rightarrow D(X)$ induces an equivalence of categories \tilde{C} and $\tilde{C}(X)$ for any sequence of blow-ups $\pi : X \rightarrow \mathbf{CP}^2$. From this we construct an action of the Cremona group on the category \tilde{C} by equivalences of category.

Then we verify, that this action preserves the isomorphism class of object P , so the Cremona group would act by outer automorphisms on A .

At last we compute this action on elements $x, y \in A$ and verify that it coincides with our explicit formulas.

To start with, we show how this construction works for the category D without taking left orthogonal to the structure sheaf, so we get an action

of the Cremona group on the category $C = D/D^1$. In this case we just recover the action of Cremona group on the field $K = \mathbf{C}(\mathbf{CP}^2)$. The functor $\tilde{C} \rightarrow C$ induced from inclusion $\tilde{D} \subset D$ corresponds to the "commutativization" morphism $comm : A \rightarrow K$.

4.3. Action of the Cremona group on K through derived categories. $\mathbf{C}(X)$ the field of rational functions on X . $D^b(\mathbf{C}(X) - mod)$ - bounded derived category of finitely generated modules over it.

Lemma 4.1. *The functor of restriction to a generic point induces an equivalence of triangulated categories $C(X) = D(X)/D^1(X)$ and $D^b(\mathbf{C}(X) - mod)$.*

Proof. Let $\eta(F)$ be a restriction of a coherent sheaf F from X to a generic point. Clearly $\eta(F)$ is a finite rank vector space over $\mathbf{C}(X)$. It induces a functor $\eta : D(X) \rightarrow D^b(C(X) - mod)$. Complexes of sheaves with the support of codimension ≥ 1 are exactly the ones that become acyclic after applying this functor. So we have a functor from the quotient category $C(X) = D(X)/D^1(X)$ to $D^b(\mathbf{C}(X) - mod)$. All the rank 1 sheaves are isomorphic in $C(X)$, let us denote by O it's isomorphism class. All the vector bundles on X are obtained by extensions from rank 1 sheaves, so the triangulated subcategory of $C(X)$ generated by cones and shifts of O contains images of vector bundles and thus is equivalent to $C(X)$.

If Z is a closed sub-scheme of X and $i : U \hookrightarrow X$ is a complement of Z , then $D(X)/D_Z(X)$ is equivalent to $D(U)$, where the equivalence is induced by a restriction functor $i^* : D(X) \rightarrow D(U)$. It's a well-known fact. In particular if U is affine then there is a commutative algebra $Q = O_U(U)$ and $D(U)$ is equivalent to $D^b(Q - mod)$. In particular $Hom_{D(X)/D_Z(X)}^i(O_U, O_U) = 0$ for $i \neq 0$ and $Hom_{D(X)/D_Z(X)}^0(O, O) = O_U(U)$.

By definition of the quotient category

$$Hom^i C(X)(A, B) = \varinjlim_f Hom_{D(X)}^i(A, F)$$

The limit in the right hand side is taken over morphisms $f : B \rightarrow F$ such that $Cone(f) \in D^1(X)$. For a given f let Z_1 be a support of $Cone(f)$. It is a closed proper sub-scheme of X . We choose a closed sub-scheme Z , such that $Z_1 \subset Z \subset X$ and the complement of Z in X is affine. So we may rewrite the formula as follows:

$$Hom^i C(X)(A, B) = \varinjlim_Z \varinjlim_g Hom_{D(X)}^i D(A, F)$$

First limit is over closed sub-schemes Z with affine complement, second limit is over g such that $Cone(g)$ is supported in Z . But now the second limit equals to $Hom_{D(X)/D_Z(X)}^i(F, G) = Hom_{D(U)}^i(F, G)$. In particular $Hom_{C(X)}^i(O, O) = 0$ for $i \neq 0$ and $Hom_{C(X)}^0(O, O) = \varinjlim_Z O_U(U) = \mathbf{C}(X)$. So the functor induced from restriction to generic point $C(X) = D(X)/D^1(X) \rightarrow D^b(\mathbf{C}(X) - mod)$ is fully faithful on O . And because every

object of $C(X)$ is a direct sum of shifts of O , it is fully faithful, thus an equivalence. \square

So we recover a field $K = \mathbf{C}(X)$ of rational functions from derived category of coherent sheaves. Let us now recover the action of the group of birational automorphisms on it. Suppose $\pi : X \rightarrow Y$ is a blow-up with a smooth center.

Lemma 4.2. . *The functor between quotient categories $\Psi : C(Y) \rightarrow C(X)$ induced by $L\pi^*$ is an equivalence and is a composition of equivalences $C(X) \xleftarrow{\eta_X^*} D^b(K\text{-mod}) \xrightarrow{\eta_Y^*} C(Y)$.*

Proof. If η_X is an embedding of a generic point in X and η_Y in Y , then clearly $\pi \circ \eta_X = \eta_Y$, so in the derived categories of coherent sheaves we have a functor isomorphism $\eta_X^* \circ L\pi^* = \eta_Y^*$.

Let E be an exceptional curve of this blow-up, then we have a semi-orthogonal decomposition $D(X) = \langle L\pi^*D(Y), O_{|E} \rangle$ and also $D^1(X) = \langle L\pi^*D^1(Y), O_{|E} \rangle$. So we can use that the quotient triangulated category $\langle A, B \rangle / B$ is equivalent to A . It follows that $D(X) / \langle O_{|E} \rangle \simeq L\pi^*D(Y)$, so

$$D(X) / D^1(X) = (D(X) / \langle O_{|E} \rangle) / L\pi^*D(Y) \simeq L\pi^*D(Y) / L\pi^*D^1(Y).$$

So $L\pi^*$ induces an equivalence between $C(Y)$ and $C(X)$. \square

It means that the quotient categories are independent of the rational surface, thus implying that the Cremona group acts on both. All line bundles placed in degree 0 become isomorphic in this category $C(X)$, so the Cremona group preserves the isomorphism class of line bundles and acts in a natural way on the endomorphism ring of this class, which is K . It's easy to verify that this action coincides with the natural action of $Cr = \text{Aut}(\mathbf{C}(x, y) / \mathbf{C})$ on $K = \mathbf{C}(x, y)$.

4.4. Construction of the birational invariant. Here we make a general construction, which to a smooth proper algebraic manifold X associates a triangulated category \tilde{C} invariant under birational isomorphisms.

Let $\tilde{D}(X)$ be a full subcategory of $D^b(\text{Coh}(X))$ consisting of objects left orthogonal to O_X , i.e. complexes of sheaves F^* , such that $R\text{Hom}_{D(X)}(F^*, O_X)$ is acyclic. It is a triangulated category and it may be empty for some manifolds.

Consider its full triangulated subcategory $\tilde{D}^1(X)$ consisting of the complexes of sheaves that are left orthogonal to O_X and have a positive codimension of support, it means the cohomology of the complex have support on the subvarieties of dimension strictly less than the dimension of X .

Define $\tilde{C}(X)$ as a quotient of triangulated categories $\tilde{C}(X) = \tilde{D}(X) / \tilde{D}^1(X)$ in a sense of Verdier[5].

So for any object Y in $\tilde{D}(X)$ let Q_Y be a category of morphisms $f : Y \rightarrow Z$, such that $\text{Cone}(f)$ is isomorphic to an object of $\tilde{D}^1(X)$. Then by results of Verdier that we reproduce from formula (12.1) of [2] we compute the morphisms in a quotient category this way:

$$(4.1) \quad \text{Ext}_{\tilde{D}(X)/\tilde{D}^1(X)}^i(U, Y) = \lim_{\substack{\longrightarrow \\ (Y \rightarrow Z) \in Q_Y}} \text{Ext}_{\tilde{D}(X)}^i(U, Z)$$

Theorem 4.1. *If smooth X and Z are birationally equivalent, then $\tilde{C}(X)$ and $\tilde{C}(Z)$ are equivalent as triangulated categories.*

Proof. In virtue of the results of [1](weak factorization theorem) any birational automorphism may be decomposed as a sequence of blow-ups and blow-downs with smooth centers. It means that it is enough to prove our theorem for such kind of morphisms.

Let $\pi : Z \rightarrow X$ be a blow-up of X with a smooth center Y . We have a pair of adjoint functors $(L\pi^*, R\pi_*$ on $D(X), D(Z)$ and moreover $L\pi^*O_X = O_Z$ and $R\pi_*O_Z = O_X$.

Let us denote $D_0(Z)$ a full subcategory of $D(Z)$ consisting of objects F such that $R\pi_*F$ is acyclic, and let $L\pi^*D(X)$ a full subcategory of $D(Z)$ consisting of images of $L\pi^*$.

- Lemma 4.3.** (1). $D_0(Z)$ and $L\pi^*D(X)$ are triangulated categories.
(2). The natural transformation $F \rightarrow R\pi_*L\pi^*F$ in $D(X)$ is an isomorphism.
(3). $L\pi^*$ induces an equivalence of triangulated categories $D(X)$ and $L\pi^*D(X)$.
(4). $L\pi^*D(X)$ is left orthogonal to $D_0(Z)$, i.e. $R\text{Hom}_{D(Z)}(L\pi^*F, G)$ is acyclic.
(5). For any object $F \in D(X)$ there is an exact triangle $F_1 \rightarrow F \rightarrow F_0$, where $F_0 \in D_0(Z)$, $F_1 \in L\pi^*D(X)$, and moreover such a triangle is unique up to isomorphism. F_1 is isomorphic to $L\pi^*R\pi_*F$.
(6). $D_0(Z)$ is supported on exceptional divisor of the blow-up.

Proof. (1) trivial. (2) is a local statement, but locally on X any object of $D(X)$ has a free bounded resolution by a structure sheaf, for which the map $O_X \rightarrow R\pi_*L\pi^*O_X$ is an isomorphism. (3) follows from projection formula $R\text{Hom}_{D(Z)}(L\pi^*F, L\pi^*G) = R\text{Hom}_{D(X)}(F, R\pi_*L\pi^*G)$ and (2). (4) also follows from the projection formula. The triangle in (5) is constructed by completing a natural map $L\pi^*R\pi_*F \rightarrow F$ to an exact triangle, map coming from the adjunction of identity morphism $R\pi_*F \rightarrow R\pi_*F$. Actually F_1 represents a functor $X \mapsto \text{Hom}_{L\pi^*D(X)}(X, F)$ and F_0 represents a functor $X \mapsto \text{Hom}_{D_0(Z)}(F, X)^*$. (6) follows from the definition $R\pi_*$, because π is an isomorphism outside an exceptional divisor. \square

A concise way to summarize lemma is the statement that $D(Z)$ admits a semi-orthogonal decomposition $D(Z) = \langle D_0(Z), L\pi^*D(X) \rangle$. Actually

we have a stronger result proved at Proposition 3.4 of [4]: there is a semi-orthogonal decomposition $D(Z) = \langle D(Y)_{-r+1}, \dots, D(Y)_1, L\pi^*D(X) \rangle$, where r is a codimension of Y in X , but we would not use it here.

We may also introduce the full triangulated subcategory $D_1(Z) \subset D$, which consists of objects $K_Z^{-1} \otimes^L F$ for $F \in D_0(Z)$. Because of Serre duality it is a left orthogonal to $L\pi^*D(X)$ and we have a semi-orthogonal decomposition $D(Z) = \langle L\pi^*D(X), D_1(Z) \rangle$.

Lemma 4.4. (1). $D_1(Z)$ is full triangulated subcategory of $\tilde{D}^1(Z)$.
 (2). There are semi-orthogonal decompositions

$$\begin{aligned}\tilde{D}(Z) &= \langle L\pi^*\tilde{D}(X), D_1(Z) \rangle \\ \tilde{D}^1(Z) &= \langle L\pi^*\tilde{D}^1(X), D_1(Z) \rangle\end{aligned}$$

Proof. Objects in $D_1(Z)$ are left orthogonal to $L\pi^*D(X)$ and in particular to $O_Z = L\pi^*O_X$. The support of an object of $D(Z)$ doesn't change when we tensor it with a linear bundle, so the elements of $D_1(Z)$ are supported on an exceptional divisor of blow-up, because the elements of $D_0(Z)$ are, as we see from the last statement of the previous lemma. \square

This lemma implies the statement of the theorem, because to obtain

$$\tilde{C}(Z) = \tilde{D}(Z)/\tilde{D}^1(Z) = \langle L\pi^*\tilde{D}(X), D_1(Z) \rangle / \langle L\pi^*\tilde{D}^1(X), D_1(Z) \rangle$$

we may first quotient $\tilde{D}(Z)$ by its full subcategory $D_1(Z)$, so we will have an equivalence

$$\tilde{D}(Z)/\tilde{D}^1(Z) \rightarrow L\pi^*\tilde{D}(X)/L\pi^*\tilde{D}^1(X)$$

And the latter is equivalent to $\tilde{C}(X)$, because $L\pi^*$ is fully faithful functor. \square

4.5. General statements about quotients of dg-categories. For convenience of the reader, let us summarize few facts concerning quotients of dg-categories. Our main reference would be [2].

Suppose we have a dg-algebra A^\bullet over a field k with a unit. We can see it as a dg-category \mathcal{A} with one object P and the endomorphism ring $\text{Hom}_{\mathcal{A}}(P, P) = A^\bullet$. Then from this dg-category with one object we construct a pre-triangulated category \mathcal{A}^{pre-tr} (cf. [D, 2.4]), which has the formal sums $B = (\oplus_{j=1}^n P^{\oplus m_j}[k_j], q)$ as objects. Here q stands for a homogeneous degree one matrix $q_{ij} \in \text{Hom}_{\mathcal{A}}^1(P^{\oplus m_j}, P^{\oplus m_i})[k_i - k_j] = \text{Mat}_{m_i \times m_j}(A^{k_i - k_j + 1})$. We ask that $q_{ij} = 0$ for $i \geq j$ and $dq + q^2 = 0$. The homomorphism between the formal sums $B = (\oplus_{j=1}^n P^{\oplus m_j}[k_j], q)$ and $B' = (\oplus_{j=1}^{n'} P^{\oplus m'_j}[k'_j], q')$ is given by matrices $f = (f_{ij})$ where

$$f_{ij} \in \text{Hom}(P^{\oplus m_j}, P^{\oplus m'_i})[k'_i - k_j] = \text{Mat}_{m'_i \times m_j}(A^{k'_i - k_j})$$

The differential on homomorphism groups is $df = (df_{ij}) + q'f + (-1)^l f q$, where $l = \deg(f_{ij})$. Define the triangulated category \mathcal{A}^{tr} as the homotopy

category of \mathcal{A}^{pre-tr} , which means $Hom_{\mathcal{A}^{tr}}(X, Y) = H^0(Hom_{\mathcal{A}^{pre-tr}}(X, Y))$. The category \mathcal{A}^{pre-tr} contains cones. Let us consider $M \in Hom_{\mathcal{A}}(P^m, P^m)$ a $m \times m$ -matrix with coefficients in A of degree 0. An element $Cone(M)$ would be an object $(P^m \oplus P^m[1], q)$, where $q_{21} = M[-1]$ and $q_{11} = q_{12} = q_{22} = 0$. Naturally this element goes to the cone of a morphism M in the triangulated category. In the set up of [2] if \mathcal{B} is a full dg-subcategory of \mathcal{A} , then the quotient dg-category \mathcal{A}/\mathcal{B} has the same objects as \mathcal{A} and for every object $X \in \mathcal{B}$ it has an additional morphism $u_X \in Hom_{\mathcal{A}/\mathcal{B}}^{-1}(X, X)$ with differential $d(u_X) = id_X$. The resulting homotopy category $(\mathcal{A}/\mathcal{B})^{tr}$ is equivalent to the quotient $\mathcal{A}^{tr}/\mathcal{B}^{tr}$.

4.6. Localization in dg-sense. Suppose we have a closed morphism in dg-category \mathcal{A} . It will give some morphism in the homotopy category \mathcal{A}^{tr} and we want to understand what does it mean to invert this morphism. It corresponds to taking a quotient $\mathcal{A}/Cone(a)$, so that $(\mathcal{A}/Cone(a))^{tr}$ is a localization of \mathcal{A}^{tr} at a .

Let us now consider a dg-category \mathcal{A} with one object P , where $Hom_{\mathcal{A}}(P, P) = A^\bullet$ is a dg-algebra concentrated in non-positive degrees. Let $Q = Cone(a) \in \mathcal{A}^{pre-tr}$ for some element $a \in A^0$ of degree 0. We want to consider a quotient of dg-category $D_1 = \langle P, Q \rangle$ which has two objects P, Q by dg-subcategory $D_2 = \langle Q \rangle$ with one object Q . Both D_1 and D_2 are seen as full subcategories of \mathcal{A}^{pre-tr} .

Lemma 4.5. *Quotient dg-category $D_1/D_2 = \langle P, Q \rangle / \langle Q \rangle$ is equivalent to a dg-category with one object P and $End_{D_1/D_2}(P) = B$ - a dg-algebra, concentrated in non-positive degrees. Moreover $H^0(B)$ is a localization of $H^0(A)$ at a .*

Proof. First let us write homomorphism groups in the category D_1 :

$$\begin{aligned} End_{D_1}(P) &= A^\bullet \\ Hom_{D_1}(P, Q) &= A^\bullet \oplus A^\bullet[1] = A^\bullet e_0 \oplus A^\bullet e_{-1} \\ Hom_{D_1}(Q, P) &= A^\bullet \oplus A^\bullet[-1] = A^\bullet f_0 \oplus A^\bullet f_1 \\ End_{D_1}(Q) &= A^\bullet[1] \oplus A^\bullet \oplus A^\bullet \oplus A^\bullet[-1] = A^\bullet e_0 f_1 \oplus A^\bullet e_0 f_0 \oplus A^\bullet e_{-1} f_1 \oplus A^\bullet e_{-1} f_0 \end{aligned}$$

We would also have the following relations: $f_1 e_0 = f_0 e_{-1} = 0$, $f_1 e_{-1} = f_0 e_0 = 1$, $id_Q = e_0 f_0 + e_{-1} f_1$. Lower indices indicate the degrees of homomorphisms. Also all e_{-1}, e_0, f_0, f_1 commute with A^\bullet .

On these spaces there would be a differential: $de_{-1} = ae_0$, $df_0 = -f_1 a$.

The procedure described in [2] suggests that the quotient category would have one more homomorphism $u \in End(Q)$ of degree -1 , such that $du = id_Q$. So as an A^\bullet -algebra $B = End_{D_1/D_2}(P)$ would be freely generated by four elements: $t = f_1 u e_0$, $x = f_1 u e_{-1}$, $y = f_0 u e_0$, $z = -f_0 u e_{-1}$. Of them t would be of degree 0, x, y - of degree -1 , and z of degree -2 . One verifies

the following formulas: $dt = 0$, $dx = ta - 1$, $dy = 1 - at$, $dz = ax + ya$. The elements of B of degree 0 are $A^0 < z >$ - non-commutative polynomials in z with coefficients of degree 0 in A , the elements of degree -1 are $a_1xa_2 + a_3ya_4 + a_5$, where a_i are of degree 0 and a_5 is of degree -1 . So $H^0(B) = H^0(A^\bullet)[z]/(az - 1, za - 1)$ which is a localization of $H^0(A^\bullet)$ at a , and the lemma is proved. \square

Comment. At this point it is not clear, whether the algebras that we obtain have non-trivial cohomologies in negative degrees.

For further use we should improve this lemma for cones of more general type. As before we consider $D_1 = \langle P, Q \rangle$, $D_2 = \langle Q \rangle$ full subcategories of \mathcal{A}^{pre-tr} , but now $Q = Cone(M)$ where M is a matrix over A^\bullet of homogeneous degree 0. We would prove:

Lemma 4.6. *Suppose that $M = \begin{pmatrix} 1_{k \times k} & 0 \\ 0 & a \end{pmatrix} + dM_{-1}$, $a \in A^\bullet$ is a degree 0 element. Then $End_{D_1/D_2}(P) = B$ - a dg-algebra, concentrated in non-positive degrees and $H^0(B)$ is a localization of $H^0(A^\bullet)$ at a .*

Proof. Remember that $Cone(M) = (P^{\oplus m} \oplus P^{\oplus m}[1], q = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix})$. So we would have the basis $e_0^i, e_{-1}^i, i = 1..k+1$ of $Hom_{D_1}(P, Q)$ as an A^\bullet -module.

We would for simplicity denote by the column vectors $e_0 = \begin{pmatrix} e_0^1 \\ \vdots \\ e_0^{k+1} \end{pmatrix}$, $e_{-1} =$

$\begin{pmatrix} e_{-1}^1 \\ \vdots \\ e_{-1}^{k+1} \end{pmatrix}$. Moreover we can multiply these elements by A^\bullet from the left as

well as from the right and these multiplications commute. $de_{-1} = Me_0$. Then we can pick a dual basis of $Hom_{D_1}(Q, P)$: $f_0^i, f_1^i, i = 1..k+1$, and similarly denote row vectors $f_0 = (f_0^1, \dots, f_0^{k+1})$, $f_1 = (f_1^1, \dots, f_1^{k+1})$. Then $df_0 = -f_1M$. We would have $f_1^i e_0^j = f_0^i e_{-1}^j = 0, f_1^i e_{-1}^j = f_0^i e_0^j = \delta_{ij}, id_Q = \sum_{i=0}^{k+1} (e_0^i f_0^i + e_{-1}^i f_1^i)$.

Again the quotient category is obtained adding element u of degree -1 to the space $End(Q)$ such that $du = id_Q$. So $B = End_{D_1/D_2}(P)$ has no elements in positive degree. In degree 0 it is freely generated by $f_1^i u e_0^j$ as A^0 -algebra. So B^0 as an algebra is isomorphic to $A^0 < u_{ij} >$ - non-commutative polynomials over A^0 and u_{ij} maps through isomorphism to $f_1^i u e_0^j$. Further notice that $d(f_0^i u e_0^j) = (df_0^i) u e_0^j + f_0^i (du) e_0^j = -(\sum_k m^{ik} f_1^k u e_0^j) + f_0^i Id_Q e_0^j = -(\sum_k d m_{-1}^{ik} (f_1^k u e_0^j)) - (m^{ii} - d m_{-1}^{ii}) f_1^i u e_0^j + \delta_{ij}$, where m_{-1}^{ij} stand for entries of the matrix M_{-1} . We know that $d(f_1^k u e_0^j) = 0$, so modulo the border $\delta_{ij} = (m^{ii} - d m_{-1}^{ii}) f_1^i u e_0^j$. But $(m^{ii} - d m_{-1}^{ii}) = 1$ for $i \neq k+1$, so in $H^0(B)$ we have $\delta_{ij} = f_1^i u e_0^j$ for $i \neq k+1$. The similar calculation of $d(f_0^i u e_{-1}^j)$

will show that $\delta_{ij} = f_1^i u e_0^j$ for $j \neq k+1$. So we deduce that $H^0(B)$ is generated by $t = f_1^{k+1} u e_0^{k+1}$ over $H^0(A^\bullet)$. The border will be generated by $d(f_0^{k+1} u e_0^{k+1}) = -at + 1$ and $d(f_1^{k+1} u e_{-1}^{k+1}) = -ta + 1$. So $H^0(B)$ is a localization of $H^0(A^\bullet)$ at a , which proves the lemma. \square

4.7. Computation of $\tilde{C}(\mathbf{CP}^2) = \tilde{D}/\tilde{D}^1$. It is known that a derived category D of coherent sheaves on \mathbf{CP}^2 is generated by $O, O(1), O(2)$. Consider a subcategory \tilde{D} which is left orthogonal to O , it is generated by $O(1)$ and $O(2)$. Denote $V = R\text{Hom}_D(O(1), O(2)) = \text{Hom}_{\mathbf{CP}^2}(O(1), O(2))$ - a 3-dimensional vector space.

Denote $P_1 = O(1), P_2 = O(2)$, then $\text{Hom}_D(P_1, P_2) = V$ and all the other higher hom's between these objects vanish. $\text{End}(P_1) = \text{End}(P_2) = \mathbf{C}$. Any object in D may be written as a direct sum of shifts of the complexes $P_1^m \rightarrow P_2^n$ and the arrow is given by an $n \times m$ matrix with elements in V .

D^1 is a full subcategory of D , consisting of complexes of coherent sheaves with the dimension of support at most 1. Then \tilde{D}^1 is the intersection of D^1 and \tilde{D} , more precisely it is full subcategory of D , consisting of objects left orthogonal to O and with a dimension of support at most 1. The condition of left orthogonality to O means that objects in \tilde{D}^1 are direct sums of shifts of $\text{Cone}(M)$, where $M : O(1)^{\oplus m} \rightarrow O(2)^{\oplus n}$. The condition to have support of dimension at most 1 means, that the map given by M is generically invertible, which is exactly the case when $m = n$ and the commutative determinant of M is non-trivial.

Choose a non-zero element $e \in V$. First we would prove that $\tilde{D}/\text{Cone}(e) \simeq D^b(R\text{-mod})$, where $D^b(R\text{-mod})$ is the bounded derived category of complexes R -modules of finite type.

We would proceed in a dg-setting, so we consider a dg-category Q with two objects P_1, P_2 with $\text{Hom}(P_1, P_2) = V$ - 3-dimensional of degree 0, $\text{Hom}(P_2, P_1) = 0$, endomorphism rings of both objects are one-dimensional and are generated by identity morphisms. This category corresponds to a Kronecker quiver with three arrows. The associated derived category Q^{tr} is equivalent to \tilde{D} . First we would quotient it by an object $\text{Cone}(e : P_1 \rightarrow P_2)$, where e is some non-trivial morphism. Denote by Q_1 a full dg-subcategory of Q^{pre-tr} consisting of objects $P_1, P_2, \text{Cone}(e)$. Let us also denote by \mathcal{R} a dg-category with one object P and $\text{Hom}_{\mathcal{R}}(P, P) = R = \mathbf{C} \langle x, y \rangle$ - an algebra of non-commutative polynomials on x, y of degree 0.

Lemma 4.7. *Dg-categories \mathcal{R} and $Q_1/\text{Cone}(e)$ are quasi-equivalent.*

Corollary. Proposition 2.5[2] then imply that \mathcal{R}^{pre-tr} and $(Q_1/\text{Cone}(e))^{pre-tr}$ are quasi-equivalent, which means that $D^b(R\text{-mod})$ and $(Q_1/\text{Cone}(e))^{tr}$

are equivalent as triangulated categories.

Proof. We understand $\text{Cone}(e)$ as a $P_2 \oplus P_1[1]$ with a differential $e : P_1 \rightarrow P_2$ and symbolically write it as a column $\begin{pmatrix} P_2 \\ P_1[1] \end{pmatrix}$, so the endomorphism dg-algebra $\text{End}_{Q_1}(\text{Cone}(e))$ looks like

$$\begin{pmatrix} \text{Hom}(P_2, P_2) & \text{Hom}(P_1, P_2)[-1] \\ \text{Hom}(P_2, P_1)[1] & \text{Hom}(P_1, P_1) \end{pmatrix} = \begin{pmatrix} k & V[-1] \\ 0 & k \end{pmatrix}$$

k means 1-dimensional vector spaces of degree 0, and $V = \text{Hom}(P_1, P_2)$. The differential is the following: $d : \begin{pmatrix} x & v \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} 0 & (x-y)e \\ 0 & 0 \end{pmatrix}$. The multiplication is coming from the matrix multiplication. Denote this dg-algebra by R^\bullet .

Choose a 2-dimensional vector subspace $V' \subset V$, such that $V' \oplus ke = V$ and let a, b be it's basis. By construction of [2] dg-category $Q_1/\text{Cone}(e)$ is obtained from Q_1 by adding a morphism $\eta \in \text{Hom}^{-1}(\text{Cone}(e), \text{Cone}(e))$ such that $d\eta = \text{id}_{\text{Cone}(e)}$. We also have

$$\text{Hom}_{Q_1}(P_2, \text{Cone}(e)) = \begin{pmatrix} k \\ 0 \end{pmatrix}$$

$$\text{Hom}_{Q_1}(\text{Cone}(e), P_2) = \begin{pmatrix} k & V[-1] \end{pmatrix}$$

They are respectively left and right R^\bullet -dg-modules.

$$\text{Hom}_{Q_1/\text{Cone}(e)}(P_2, P_2) = \bigoplus_{n=0}^{\infty} \text{Hom}_{(n)}(P_2, P_2)$$

$$\text{Hom}_{(n)}(P_2, P_2) = \text{Hom}_{Q_1}(\text{Cone}(e), P_2) \otimes \eta \otimes R^\bullet \otimes \eta \otimes \dots R^\bullet \otimes \eta \otimes \text{Hom}_{Q_1}(P_2, \text{Cone}(e))$$

n is a number of times that η shows up. In particular we see that this is a dg-algebra concentrated in non positive degrees. The degree 0 part is:

$$k \oplus V\eta k \oplus V\eta V\eta k \oplus \dots = T(V)$$

$T(V)$ is a tensor algebra of V .

This allows us to construct a DG functor $F : \mathcal{R} \rightarrow Q_1/\text{Cone}(e)$. We define $F(P) := P_2$, and $F(x) = a\eta i$, $F(y) = b\eta i$, where i is a generator of $\text{Hom}_{Q_1}(P_2, \text{Cone}(e))$. Remind that $\text{Hom}_{\mathcal{R}}(P, P) = R = \mathbf{C} \langle x, y \rangle = T(W)$, where W is a vector space spanned by x, y . So F sends $T(W)$ to $T(V') \subset T(V) = \text{Hom}_{Q_1/\text{Cone}(e)}(P_2, P_2)$.

In the dg-category $Q_1/\text{Cone}(e)$ objects P_1 and P_2 are homotopy equivalent, because $e \in \text{Hom}_{Q_1/\text{Cone}(e)}^0(P_2, P_1)$ and $j\eta i \in \text{Hom}_{Q_1/\text{Cone}(e)}^0(P_1, P_2)$ are inverse of each other up to a homotopy (j is a generator of $\text{Hom}_{Q_1}(\text{Cone}(e), P_1)$). So the functor $Ho(F)$ is essentially surjective and to conclude the lemma we need just to prove that F induces a quasi-isomorphism between $\text{End}_{\mathcal{R}}(P)$

and $End_{Q_1/Cone(e)}(P_2)$, in other words we have to prove that the cohomology of the latter complex is exactly $T(V')$.

Introduce a dg-subalgebra $R_0^\bullet \subset R^\bullet$ consisting of $\begin{pmatrix} k & ke \\ 0 & k \end{pmatrix}$ and let $R' \subset R^\bullet$

be a submodule $\begin{pmatrix} 0 & V' \\ 0 & 0 \end{pmatrix}$, so that we have a decomposition $R^\bullet = R_0^\bullet \oplus R'$.

Introduce a dg-subalgebra $T_\eta(R_0^\bullet) \subset Hom_{Q_1/Cone(e)}(P_2, P_2)$ by the following formula:

$$T_\eta(R_0^\bullet) = \bigoplus_{n=0}^{\infty} \eta \otimes R_0^\bullet \otimes \eta \dots R_0^\bullet \otimes \eta$$

n is the the number of times the R_0^\bullet shows up.

The cohomology of R_0^\bullet is one dimensional and generated by the identity matrix Id . Let us introduce a subspace $T_\eta(Id) \subset T_\eta(R_0^\bullet)$ defined as follows:

$$T_\eta(Id) = k\eta \oplus k\eta \otimes Id \otimes \eta \oplus k\eta \otimes Id \otimes \eta \otimes Id \otimes \eta \oplus \dots$$

Let $C_0 = \begin{pmatrix} k & ke \end{pmatrix}$, so that we have decompositions:

$$\begin{aligned} Hom_{Q_1}(Cone(e), P_2) &= C_0 \oplus R' \\ R^\bullet &= R_0^\bullet \oplus R' \end{aligned}$$

Substituting this to the formula(4.7) we get

$$Hom_{Q_1/Cone(e)}(P_2, P_2) = \bigoplus_{m=0}^{\infty} Hom'_m(P_2, P_2)$$

$$Hom'_m(P_2, P_2) = (R' \oplus C_0) \otimes T_\eta(R_0^\bullet) \otimes R' \otimes T_\eta(R_0^\bullet) \otimes \dots \otimes R' \otimes T_\eta(R_0^\bullet) \otimes ki$$

Or in a more concise way

$$Hom_{Q_1/Cone(e)}(P_2, P_2) = (R' \oplus C_0) \otimes T_\eta(R_0^\bullet) \otimes T(R' \otimes T_\eta(R_0^\bullet)) \otimes ki$$

Observe here that as a complex it is a direct sum of tensor products of direct summands of complexes of the type $C_0 \otimes T_\eta(R_0^\bullet) \otimes R'$ and $R' \otimes T_\eta(R_0^\bullet) \otimes R'$. There is a well-defined filtration on the number of appearances of η , so we can compute the cohomology of these complexes using the spectral sequence with respect to these filtration. First term of the first spectral sequence is $H^\bullet(C_0) \otimes T(H^*(R^\bullet)) \otimes R'$, H^* - is cohomology. C_0 is acyclic, so $H^*(C_0) = 0$ and $C_0 \otimes T_\eta(R_0^\bullet) \otimes R'$ is acyclic. First term of the second spectral sequence is $R' \otimes T_\eta Id \otimes R'$ and the differential is given by $d(\eta) = 1$. η is odd, so the cohomology of this complex is $R'\eta R'$. We deduce from here that the cohomology of $Hom_{Q_1/Cone(e)}(P_2, P_2)$ is equal to $\bigoplus_{k=0}^{\infty} R'\eta R'\eta \dots R'\eta ki = T(R') = T(V')$. It's exactly what we needed to prove. \square

Next we have to quotient the triangulated category $D^b(R - mod)$ by the elements $Cone(M)$, where $M : R^{\oplus k} \rightarrow R^{\oplus k}$ is given by the matrices with entries in vector space $V = \langle 1, x, y \rangle$ and non-zero commutative determinant, because this objects generate the image of category \tilde{D}^1 in the quotient $\tilde{D}/Cone(e)$.

Let us proceed by induction on the size of matrices M : denote Mat_k a set of $k \times k$ matrices with coefficients in a 3-dimensional vector space $V = \langle 1, x, y \rangle$ with non-negative commutative determinant. Actually we want to invert these matrices. On each step we would obtain a dg-algebra A_k^\bullet concentrated in non-positive degrees, such there the cohomology of this algebra would be the endomorphisms of the object P in a quotient category $\mathcal{A}^{tr} / \langle Cone(M) | M \in Mat_k \rangle$.

Theorem 4.2. . *There exist dg-algebras A_k^\bullet , $k = 0, 1, 2, \dots$ concentrated in non-positive degrees, such that:*

1. $A_0^\bullet = R = \mathbf{C} \langle x, y \rangle$ in degree 0;
2. For a triangulated category $D_k = D^b(R - mod) / Cone(M) | M \in Mat_k$ we have $RHom_{D_k}(P, P) \cong A_k^\bullet$;
3. $H^0(A_{k+1})$ is a localization of $H^0(A_k)$ at some set outside the commutator ideal;
4. All matrices in Mat_k are invertible over $H^0(A_k)$

Corollary 1. This procedure gives us a dg-algebra $A^\bullet = \varinjlim A_k$, such that for the triangulated category $\tilde{C} = \tilde{D} / \tilde{D}^1$ we have an equivalence $RHom_{\tilde{C}}(P, P) \cong A^\bullet$.

Corollary 2. $H = \varinjlim H^0(A_k)$ is a non-commutative algebra and may be described as a consecutive localization of R at some sets, which are the elements not lying in the commutator ideal.

Proof. We proceed by induction: let's put $A_0^\bullet = R$ concentrated in degree 0. For $k = 0$ there is no statement to prove. Let us prove the theorem for $k + 1$: take a matrix $M \in Mat_{k+1}$, as it has non-trivial commutative determinant, we can find sub-matrix M' of size $k \times k$ that would also have non-trivial determinant. This matrix would be invertible over $H^0(A_k^\bullet)$, so let M'_1 be the lift of it's inverse over A_k^\bullet . We have $M'M'_1 = 1 + dM_2$ over A_k^\bullet . So if matrix M looks like $\begin{pmatrix} M' & b \\ a & c \end{pmatrix}$, then we would multiply it from the left by matrix $\begin{pmatrix} M'_1 & 0 \\ -aM'_1 & 1 \end{pmatrix}$ and from the right by matrix $\begin{pmatrix} 1 & -M'_1b \\ 0 & 1 \end{pmatrix}$. So the resulting matrix would be $M_0 = \begin{pmatrix} 1_{k \times k} & 0 \\ 0 & q \end{pmatrix} + dM_3$, where $q = -aM'_1b + c \in A_k^\bullet$ and M_3 is some $(k+1) \times (k+1)$ matrix of degree -1 defined over A_k^\bullet .

Now we can apply the previous lemma, which would say that the quotient of $(A_k^\bullet)^{pre-tr}$ by $Cone(M)$ is dg-equivalent to \mathcal{B}_M^{pre-tr} , where B_M is some dg-algebra, concentrated in non-positive degrees and $H^0(B)$ is a localization of $H^0(A)$ at q . When we quotient by a set of matrices, the procedure of [2] takes the direct limit of dg-algebras, so we would obtain $A_{k+1}^\bullet = \varinjlim_M B_M$

and $H^0(A_{k+1}^\bullet)$ would be just a localization of $H^0(A_k^\bullet)$ at the set of elements obtained as q in the beginning of the proof. After the commutativization q goes to the determinant of the matrix M divided by the determinant of the matrix M' , so definitely it is non-zero, so it lies outside the commutator ideal. \square

As a consequence of the theorem we obtain a non-commutative algebra $H = H^0(A^\bullet) = RHom_{\tilde{C}(\mathbf{CP}^2)}^0(P, P)$ and it is a consecutive localization of $R = \mathbf{C} \langle x, y \rangle$ at some sets outside the commutator ideal. Although that is enough to get the action of the Cremona group on the algebra A , because we have an algebra morphism $H \rightarrow A$, we can prove the more precise result:

Lemma 4.8. *The map $H = H^0(A^\bullet) \rightarrow A$ is an isomorphism.*

Proof. We can summarize the procedure of the theorem as follows: consider a matrix $M \in M_k(V)$ with values in a vector space $V = \langle 1, x, y \rangle$ and non-zero commutative determinant. Remind that M is invertible over H_k , or more precisely up to permuting the columns of M we can represent it as a product $M = UT$, where U is a lower triangular matrix with coefficients in H_{k-1} and it's diagonal elements are invertible over H_{k-1} and the last diagonal element is 1, T is an upper triangular matrix with coefficients in H_{k-1} . The elements on the diagonal of T are $(1, 1, \dots, 1, \Delta)$. As proved in the theorem, Δ is invertible in H_k . Let S denote a subset of elements of H , consisting of images of elements Δ in H_k for all matrices M with non-zero commutative determinant and of any size k . In particular S contains non-zero elements of V for $k = 1$.

Let $comm : H \rightarrow K = \mathbf{C}(x, y)$ be a natural morphism induced from the map $R = \mathbf{C} \langle x, y \rangle \rightarrow K$. We will prove that S consists of all the elements of H that are not in the kernel of $comm$.

Lemma 4.9. 1. $\Delta \in S$ implies $\Delta^{-1} \in S$.

2. $\Delta_1, \Delta_2 \in S$ implies $\Delta_1 \Delta_2 \in S$.

3. If $\Delta_1, \Delta_2 \in S$ and $comm(\Delta_1 + \Delta_2) \neq 0$ then $\Delta_1 + \Delta_2 \in S$.

Proof. Notice first that if we have a matrix $M \in Mat_{k \times k}(V)$ and an equality $UM = T$, where U is lower triangular and T is upper triangular with diagonal elements $(1, \dots, 1, \Delta)$, then for any $\lambda \in \mathbf{C}^*$ let us consider a diagonal matrix $D = diag(1, \dots, 1, \lambda)$. Then MD has also coefficients in V and non-zero determinant and $U(MD) = (TD)$, where (TD) has elements $(1, \dots, 1, \lambda\Delta)$ on diagonal. This implies that $\lambda\Delta \in S$.

Given M and decomposition $UM = T$, let us consider a matrix $M' = \begin{pmatrix} M & e(k) \\ e(k)^t & 1 \end{pmatrix}$, where $e(k)$ is a $k \times 1$ matrix with $e(k)_{i1} = 0$ for all i except $e(k)_{k1} = 1$. Clearly M' has all coefficients in V . Let Δ_M be the last diagonal element of T and consider the lower-triangular matrix U' such that it's non-zero entries are $u'_{ii} = 1$ for $i = 1..k-1$, $u'_{kk} = \Delta_M^{-1}$, $u'_{k+1k} = -1$,

$u'_{k+1k+1} = 1$. Then we have a decomposition $(U'U)M' = T'$, here U is extended to a $(k+1) \times (k+1)$ matrix by putting $u_{k+1k+1} = 1$. So we have that T' has all diagonal elements equal to 1 except $t'_{k+1k+1} = \Delta_M^{-1}$. This proves that if $\Delta \in S$ then $\Delta^{-1} \in S$, which is the assertion (1) of the lemma. Suppose we have $UM = T$ and $WN = Q$, and $t_{kk} = \Delta_1$, $q_{nn} = \Delta_2$. Let M_0 be the first $k-1$ columns of M and M_1 be the last column of M , N_0 - first $(l-1)$ columns of N and N_1 the last column of N . Let $O(k, l)$ be a $k \times l$ matrix consisting of zero's and $e(l)$ be a $l \times 1$ vector which is zero except for $e(l)_{l1} = 1$. Introduce matrices

$$P = \begin{pmatrix} M_0 & e(k) & O(k, l-1) & M_1 \\ O(l, k-1) & N_1 & N_0 & O(l, 1) \end{pmatrix}$$

$$U' = \begin{pmatrix} U & O(k, l) \\ O(l, k) & W \end{pmatrix}$$

Then we have

$$U'P = \begin{pmatrix} UM_0 & Ue(k) & O(k, l-1) & UM_1 \\ O(l, k-1) & WN_1 & WN_0 & O(l, 1) \end{pmatrix}$$

Notice that $Ue(k) = e(k)$ because U is lower triangular with the last diagonal entry equal to 1. Moreover the bottom entry of WN_1 equals to Δ_2 and the bottom entry of UM_1 is Δ_1 . Consider now a $l \times k$ matrix $W' = (O(l, k-1) \quad -WN_1)$ and a $(k+l) \times (k+l)$ matrix

$$U'' = \begin{pmatrix} 1_{k \times k} & O(k, l) \\ W' & 1_{l \times l} \end{pmatrix}$$

$W'UM_0 = 0$ and the bottom entry of $W'UM_1$ is $-\Delta_2\Delta_1$. So we see that $U''U'P$ is an upper-triangular matrix with all the diagonal entries equal to 1 except for the last one which is $-\Delta_2\Delta_1$. The matrix P has coefficients in vector space $V = \langle 1, x, y \rangle$ and $U''U'$ is lower triangular with 1's on the diagonal. So it implies that $-\Delta_2\Delta_1 \in S$, which proves the second statement of the lemma.

To prove the last statement, for a $\Delta_1 \in S$ we find the decomposition $UM = T$ where $t_{kk} = -\Delta_1 \in S$. Let also $WN = Q$ with $q_{ll} = \Delta_2$. The notations for $M_0, M_1, N_0, N_1, O(k, l), e(k)$ are as previously. Consider matrices

$$P = \begin{pmatrix} M_0 & e(k) & O(k, l-1) & M_1 \\ O(l, k-1) & e(l) & N_0 & N_1 \end{pmatrix}$$

$$U' = \begin{pmatrix} U & O(k, l) \\ O(l, k) & W \end{pmatrix}$$

$$U'P = \begin{pmatrix} UM_0 & e(k) & O(k, l-1) & UM_1 \\ O(l, k-1) & e(l) & WN_0 & WN_1 \end{pmatrix}$$

The matrix $U'P$ is upper-triangular except for one entry $(U'P)_{k+l, k} = 1$. Let us multiply this matrix to the left by U'' such that $u''_{ii} = 1$ for all

i , $u''_{k+l,k} = -1$ and other entries are zero. Then $U''U'P$ would be upper triangular with the last diagonal entry equal to $\Delta_1 + \Delta_2$. \square

Remind that H is constructed as the consecutive localization at the elements coming from determinants. It follows from lemma(4.9) that any element outside of the kernel of the map $comm : H \rightarrow K$ is invertible. So we invert all the elements outside of the commutator ideal, which is a construction of A . So we proved the lemma. \square

4.8. Action of the Cremona group on \tilde{C} . Suppose that $f \in Cr$ is an element of the Cremona group. There exists a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ \mathbf{CP}^2 & \xrightarrow{f} & \mathbf{CP}^2 \end{array}$$

π_X, π_Y - are sequences of blow-ups and g is a regular iso-morphism. By Theorem 4.1 $(L\pi_X^*)^{-1} \circ Lg^* \circ L\pi_Y^*$ is a composition of equivalences of triangulated categories, so it defines an auto-equivalence ψ_f of $\tilde{C} = \tilde{D}/\tilde{D}^1$. Moreover the composition of such auto-equivalences $\psi_f \circ \psi_g$ is non-canonically isomorphic to $\psi_{f \circ g}$, so we have a well-defined weak action of the Cremona group on \tilde{C} .

Notice that $Lg^* \circ L\pi_Y^*(O_{\mathbf{CP}^2}(1)) = g^*(\pi_Y(O_{\mathbf{CP}^2}(1)))$ is a line bundle on X left orthogonal to O_X .

Lemma 4.10. *The isomorphism class of an object $P = O(1) \in \tilde{C}$ is preserved by the action of the Cremona group.*

Proof. As previously we suppose that our birational automorphism is decomposed into blow-up $\pi : X \rightarrow \mathbf{CP}^2$ and blow-down $f : X \rightarrow \mathbf{CP}^2$ and it is given by $f \circ \pi^{-1}$. So $L = f^*O(1) = Lf^*O(1)$ is a line bundle on X , left orthogonal to O . If an automorphism is given by the element of $PGL(3, \mathbf{C})$ then we don't need blowing up and f is this regular morphism, so $Lf^*O(1) = O(1)$ and the isomorphism class is preserved. If automorphism is a quadratic transformation, then π is blow-up of \mathbf{CP}^2 at three points and E is an exceptional curve. Then $Lf^*O(1) = O(2 - E)$ is a line bundle and moreover an embedding $O(2 - E) \hookrightarrow O(2)$ induces isomorphism of $Lf^*O(1)$ and $O(2)$ which is in turn isomorphic to $O(1)$ in $\tilde{C}(X)$. \square

4.9. The coherence of two actions. Cremona group acts on \tilde{C} and fixes P , so it acts by outer automorphisms on $H^k(End(P))$. And as we calculated in Lemma(4.8) $H^0(End(P)) = A$ - an algebra constructed previously(2). This equality is also specified up to inner conjugation. So we obtain an action of Cr on A by outer automorphisms.

Lemma 4.11. . *The action of the Cremona group on $A = H^0(\text{End}_{\tilde{D}/\tilde{D}^1}(P))$ coincides with the one constructed by explicit formulas(3).*

Proof. Our isomorphism between A and $H^0(\text{End}_{\tilde{D}/\tilde{D}^1}(P))$ depended on the choice of the basis of vector space $H^0(\mathbf{CP}^2, O(1)) = V = \langle e, a, b \rangle$. We used e to identify $O(1)$ and $O(2)$ and we denoted $e^{-1}a, e^{-1}b$ by $x, y \in \text{End}(P)$. Let e', a', b' be another basis, they are some linear combinations of e, a, b , so $\gamma = e^{-1}e', \alpha = e^{-1}a', \beta = e^{-1}b'$ are some linear combinations of $1, x, y$. So $x' = e'^{-1}a' = (e^{-1}e')^{-1}(e^{-1}a') = \gamma^{-1}\alpha, y' = \gamma^{-1}\beta$.

It follows that if we have an automorphism of \mathbf{CP}^2 given by $(x : y : 1) \mapsto (\alpha, \beta, \gamma)$, then it would induce an automorphism of $H^0(\text{End}_{\tilde{D}/\tilde{D}^1}(P)) = A$ given by $(x, y) \mapsto (\gamma^{-1}\alpha, \gamma^{-1}\beta)$. We should verify that this map is conjugate to the one, given by explicit formulas.

If $\alpha = ax + by + c, \beta = dy + e, \gamma = 1$, then both maps are written as $(x, y) \mapsto (ax + by + c, dy + e)$.

If $\alpha = y, \beta = x, \gamma = 1$, then both maps are given by $(x, y) \mapsto (y, x)$.

If $\alpha = x, \beta = 1, \gamma = y$, then map coming from derived category is $(x, y) \mapsto (y^{-1}x, y^{-1})$. A map defined by explicit formulas is a composition of $(x, y) \mapsto (y^{-1}x, y)$ and $(x, y) \mapsto (x, y^{-1})$, which is $(x, y) \mapsto (y^{-1}x, y^{-1})$, so both maps in question coincide.

This three types of maps generate the action of the group $GL(3, \mathbf{C})$ on A by outer automorphisms, and we've just proved that the two actions coincide. The Cremona group is known to be generated by linear automorphisms and a quadratic transformation, so we are left to verify that actions are conjugate for quadratic transformation $a : (x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$.

The explicit formula have the form $a : (x, y) \mapsto (x^{-1}, y^{-1})$, because $a \in D_1 \times D_2$.

As we noticed before, if $f : X \rightarrow Y$ is a blow-up at one point, then $D(X)$ has a semi-orthogonal decomposition $D(X) = \langle Lf^*D(Y), O_E \rangle$, where E is an exceptional curve and O_E is it's structure sheaf.

Let us now consider surface X which is a blow-up of \mathbf{CP}^2 at 3 points $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$ and denote this map by π . Let E be an exceptional curve, so it is a union of three non-intersecting rational curves. Quadratic transformation $f : (x : y : z) \mapsto (yz, xz, xy)$ is a composition $g \circ \pi^{-1}$, where g is a regular morphism $g : X \rightarrow \mathbf{CP}^2$. $Lg^*O(1) = O(2 - E), Lg^*O(2) = O(4 - 2E)$. So the basis elements $e, a, b \in \text{Hom}(O(1), O(2))$ maps by Lg^* to some elements $e', a', b' \in \text{Hom}_X(O(2 - E), O(4 - 2E))$, so to understand the action of this quadratic transformation on non-commutative algebra A we need to understand the elements $e'^{-1}a', e'^{-1}b' \in \text{End}_{\tilde{D}/\tilde{D}^1}(P)$.

Consider a sheaf $O(1 - E)$ on X . Let us prove that it is left orthogonal to O_X , $RHom(O(1 - E), O) = R\Gamma(O(E - 1))$ and we have an exact sequence of sheaves $0 \rightarrow O(-1) \rightarrow O(-1 + E) \rightarrow O(-1 + E)|_E \rightarrow 0$. Actually $O(-1 + E)$ restricted to any component E_i of E is isomorphic to $O(E_i)|_{E_i} \cong O(-1)|_{\mathbf{CP}^1}$, so it's acyclic. $O(-1)_X$ is also acyclic, so $R\Gamma(O(-1 + E)) = 0$ which proves that $O(1 - E) \in \tilde{D}(X)$.

Now in $\tilde{D}(X)$ we have the following maps:

$$\begin{aligned} u_x, u_y, u_z &\in Hom(O(1 - E), O(2 - E)) = Hom(O(1), O(2)) \\ f = xyz &\in Hom(O(1), O(4 - 2E)) \\ a', b', e' &\in Hom(O(2 - E), O(4 - 2E)) \\ i : O(1 - E) &\hookrightarrow O(1), j : O(2 - E) \hookrightarrow O(2) \end{aligned}$$

By definition: $j \circ u_x = x \circ i$, $j \circ u_y = y \circ i$, $j \circ u_z = z \circ i$. Also we have the following relations: $f \circ i = a' \circ u_x = b' \circ u_y = e' \circ u_z$, just because $a' = yz$, $b' = xz$, $c' = xy$ as elements of $Hom(O(2 - E), O(4 - 2E)) = \Gamma(O(2 - E))$.

Now we can pass to a quotient category $\tilde{C}(X) = \tilde{D}(X)/\tilde{D}^1(X)$, where all the previously described maps become invertible, because they define non-trivial maps between line bundles, so their cones have support of dimension 1. So in the quotient category we have: $a' \circ j^{-1} \circ x \circ i = e' \circ j^{-1} \circ z \circ i$ which implies $e'^{-1}a' = j^{-1} \circ zx^{-1} \circ j$ and $e'^{-1}b' = j^{-1} \circ zy^{-1} \circ j$, so this map is conjugate to the map $(x, y) \mapsto (x^{-1}, y^{-1})$, which concludes the lemma. \square

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